SOLUTION OF THE PROBLEM OF OPTIMAL CUT IN AN ELASTIC BEAM

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The Kirchhoff model of an elastic beam with a transverse cut is considered. The nonpenetration condition proposed by A. M. Khludnev is formulated at the edges of the cut. The equilibrium model of a beam with a restriction on the cut is written in the form of a variational inequality. An analytical solution is obtained with the use of the projection operator. The problem of choosing optimal cuts is formulated for the criterion of minimum opening. Conditions for determining the extremum shapes of the beam are obtained and an example of the solution of the problem is given.

Formulations of the problems of elastic bodies with cuts (cracks) are discussed, for example, in [1–3]. In this paper, the nonpenetration condition proposed by A. M. Khludnev [4, 5] is specified at the edges of the cut. The projection operator is used to obtain an analytical solution of the problem formulated in the form of a variational inequality. The problem of choosing optimal cuts for the criterion of minimum opening [3] is posed. To this end, the solution is rewritten in the form of a dependence on a continuous function that is the solution of the problem of a cut-free beam. Conditions for determining the extremum shapes of the beam are obtained. Some approaches to the approximate solution of variational inequalities for problems with restrictions are given in [6–8]. Exact solutions of the variational inequalities and the problems of optimal control can be found only in particular cases [9, 10].

Equilibrium Problem of the Beam. Let the middle line of the beam coincide with the segment $\Omega_0 = (0,1)$. There is a transverse cut at the fixed point y $(0 < y < 1)$ of the beam. The beam thickness is equal to $2h$ $(h > 0)$. We search for the function $u = (u_1, u_2)$ of the horizontal $u_1(x)$ and vertical $u_2(x)$ displacements of the points x of the middle line of the beam under the external load $f = (f_1, f_2) \in (L_2(\Omega_0))^2$ (Fig. 1). We introduce the notation $\Omega = \Omega_0 \backslash \{y\}$. We determine the main Hilbert space

$$X = \{u \in H^1(\Omega) \times H^2(\Omega), \quad u_1 = u_2 = Du_2 = 0 \quad \text{for} \quad x = 0,\ 1\}.$$

Here the boundary conditions correspond to the clamped ends of the beam and D is the differentiation operator. Into X, we introduce the scalar product $(u, v) = \langle Du_1, Dv_1\rangle + \langle D^2u_2, D^2v_2\rangle$ and the corresponding norm $\|u\|^2 = (u,u)$, where $\langle\cdot,\cdot\rangle$ denotes the integration over Ω_0. The condition that the edges do not interpenetrate has the form [4, 5]

$$[u_1] \geqslant h|[Du_2]|,$$

where $[F] = F(y+0) - F(y-0)$ denotes the jump of the function F. If $[F] = 0$, we write $F(y)$ instead of $F(y+0) = F(y-0)$. We determine the closed convex set of admissible displacements $K = \{u \in X,\ [u_1] \geqslant h|[Du_2]|\}$ and the functional of the beam energy $\Pi(v) = 0.5\|v\|^2 - \langle f, v\rangle$. The equilibrium problem of the beam is to find the minimum of $\Pi(v)$ on the set K

$$\Pi(u) = \inf_{v\in K} \Pi(v)$$

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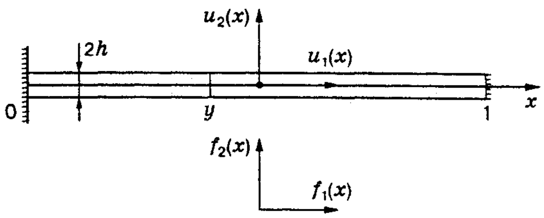

Fig. 1

or to solve the equivalent variational inequality

$$u \in K, \quad (u, v-u) \geqslant \langle f, v-u\rangle \ \ \forall v \in K. \tag{1}$$

One can easily show the uniqueness of the solution (1).

Lemma 1. *If the solution $u \in (H^2(\Omega) \times H^4(\Omega)) \cap X$ of the boundary-value problem*

$$-D^2u_1 = f_1, \qquad D^4u_2 = f_2 \quad in \quad \Omega,$$

$$[Du_1] = [D^2u_2] = 0, \quad D^3u_2(y) = 0, \quad (Du_1(y) + h^{-1}D^2u_2(y))([u_1] + h[Du_2]) = 0,$$

$$(Du_1(y) - h^{-1}D^2u_2(y))([u_1] - h[Du_2]) = 0, \quad [u_1] \geqslant h|[Du_2]|, \quad -Du_1(y) \geqslant h^{-1}|D^2u_2(y)|$$

exists, it is the unique solution of the variational inequality (1).

Proof. Integrating the equation of the boundary-value problem by parts, for arbitrary $\xi = (\xi_1, \xi_2) \in X$, we obtain

$$(u,\xi) - \langle f, \xi\rangle = \langle -D^2u_1 - f_1, \xi_1\rangle + \langle D^4u_2 - f_2, \xi_2\rangle - [Du_1 \cdot \xi_1] - [D^2u_2 \cdot D\xi_2] + [D^3u_2 \cdot \xi_2]$$
$$= -\frac{1}{2}\Big(Du_1(y) + \frac{1}{h}\, D^2u_2(y)\Big)([\xi_1] + h[D\xi_2]) - \frac{1}{2}\Big(Du_1(y) - \frac{1}{h}\, D^2u_2(y)\Big)([\xi_1] - h[D\xi_2]).$$

We set $\xi = v - u$, where $v \in K$. With allowance for the boundary conditions, we have

$$(u, v-u) - \langle f, v-u\rangle = \frac{1}{2}\Big(Du_1(y) + \frac{1}{h}\, D^2u_2(y)\Big)([u_1] + h[Du_2])$$
$$+\frac{1}{2}\Big(Du_1(y) - \frac{1}{h} D^2u_2(y)\Big)([u_1] - h[Du_2]) - \frac{1}{2}\Big(Du_1(y) + \frac{1}{h}\, D^2u_2(y)\Big)([v_1] + h[Dv_2])$$
$$-\frac{1}{2}\Big(Du_1(y) - \frac{1}{h}\, D^2u_2(y)\Big)([v_1] - h[Dv_2]) \geqslant 0 \ \ \forall v \in K.$$

Lemma 1 is proved.

We now construct an explicit solution of the problem formulated in Lemma 1. To this end, we determine the function $w \in (H^2(\Omega) \times H^4(\Omega)) \cap X$ related to f as follows:

$$-D^2w_1 = f_1, \quad D^4w_2 = f_2 \quad \text{in} \quad \Omega, \qquad Dw_1(y) = D^2w_2(y) = D^3w_2(y) = 0.$$

Given f, we find w and calculate the quantities $\varphi^+ = [w_1]+h[Dw_2]$, $\varphi^- = [w_1]-h[Dw_2]$, $\psi^+ = [w_1]+h^{-1}[Dw_2]$, and $\psi^- = [w_1] - h^{-1}[Dw_2]$. We introduce the function $\alpha \in C^\infty(\Omega)$ (Fig. 2):

$$\alpha(x) = \begin{cases} x^2/2, & x \in (0; y-0), \\ (x-1)^2/2, & x \in (y+0; 1). \end{cases}$$

The function α possesses the following properties:

$$D\alpha(x) = \begin{cases} x, & x \in (0; y-0), \\ x-1, & x \in (y+0; 1), \end{cases} \quad D^2\alpha(x) \equiv 1, \quad D^3\alpha(x) \equiv 0,\ x \in \Omega,$$

$$[D\alpha] = -1, \qquad \alpha = D\alpha = 0 \qquad \text{for} \quad x = 0,\ 1.$$

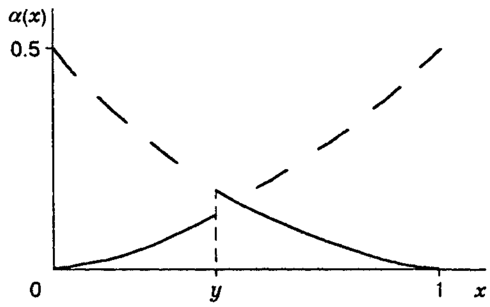

Fig. 2

We construct the function $\theta = (\theta_1, \theta_2)$, where $\theta_1 = aD\alpha$ and $\theta_2 = b\alpha$ and a and b are the constants. Obviously, $\theta \in (C^\infty(\Omega))^2 \cap X$.

Theorem 1. *The function* $u \in (H^2(\Omega) \times H^4(\Omega)) \cap X$ *that is defined by the formula*

$$u = w + \theta, \qquad \theta = (aD\alpha, b\alpha), \tag{2}$$

where

$$(a,b) = \begin{cases} (0,0) & \text{for } \varphi^+ \geqslant 0,\ \varphi^- \geqslant 0, \\ ([w_1],[Dw_2]) & \text{for } \psi^+ < 0,\ \psi^- < 0, \\ \dfrac{1}{1+h^2}(\varphi^+, h\varphi^+) & \text{for } \varphi^+ < 0,\ \psi^- \geqslant 0, \\ \dfrac{1}{1+h^2}(\varphi^-, -h\varphi^-) & \text{for } \psi^+ \geqslant 0,\ \varphi^- < 0, \end{cases} \tag{3}$$

is the solution of the variational inequality (1).

Proof. By virtue of the uniqueness of the solution (1), it is sufficient to verify the conditions for Lemma 1:

$$-D^2u_1 = -D^2w_1 - D^2\theta_1 = f_1 - aD^3\alpha = f_1 \quad \text{in} \quad \Omega,$$
$$D^4u_2 = D^4w_2 + D^4\theta_2 = f_2 + bD^4\alpha = f_2 \quad \text{in} \quad \Omega,$$
$$[Du_1] = [Dw_1] + [D\theta_1] = a[D^2\alpha] = 0, \qquad [D^2u_2] = [D^2w_2] + [D^2\theta_2] = b[D^2\alpha] = 0,$$
$$D^3u_2(y) = D^3w_2(y) + D^3\theta_2(y) = bD^3\alpha(y) = 0.$$

Since $Du_1(y) = a$, $D^2u_2(y) = b$, $[u_1] = [w_1] - a$, and $[Du_2] = [Dw_2] - b$, the last two equalities and two inequalities of the boundary-value problem of Lemma 1 take the form

$$(a + h^{-1}b)(a + hb - \varphi^+) = 0, \qquad (a - h^{-1}b)(a - hb - \varphi^-) = 0,$$
$$a + hb - \varphi^+ \leqslant 0, \quad a - hb - \varphi^- \leqslant 0, \quad a + h^{-1}b \leqslant 0, \quad a - h^{-1}b \leqslant 0.$$

These conditions are fulfilled in the following four variants:

1) $a + h^{-1}b = 0, \quad a - h^{-1}b = 0, \quad a + hb - \varphi^+ \leqslant 0, \quad a - hb - \varphi^- \leqslant 0;$

2) $a + h^{-1}b < 0, \quad a - h^{-1}b < 0, \quad a + hb - \varphi^+ = 0, \quad a - hb - \varphi^- = 0;$

3) $a + h^{-1}b < 0, \quad a - h^{-1}b = 0, \quad a + hb - \varphi^+ = 0, \quad a - hb - \varphi^- \leqslant 0;$

4) $a + h^{-1}b = 0, \quad a - h^{-1}b < 0, \quad a + hb - \varphi^+ \leqslant 0, \quad a - hb - \varphi^- = 0.$

By virtue of the relations $2\psi^+ = (1 + h^{-2})\varphi^+ + (1 - h^{-2})\varphi^-$ and $2\psi^- = (1 + h^{-2})\varphi^- + (1 - h^{-2})\varphi^+$, the variants can be written in the form

1) $a = 0, \quad b = 0, \quad \varphi^+ \geqslant 0, \quad \varphi^- \geqslant 0;$

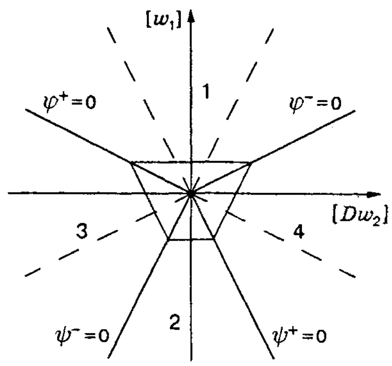

Fig. 3

$$2)\ \ a = \frac{1}{2}(\varphi^+ + \varphi^-), \quad b = \frac{1}{2h}(\varphi^+ - \varphi^-), \quad \varphi^+ < 0, \quad \varphi^- < 0;$$

$$3)\ \ a = \frac{1}{1+h^2}\varphi^+, \quad b = \frac{h}{1+h^2}\varphi^+, \quad \varphi^+ < 0, \quad \psi^- \geqslant 0;$$

$$4)\ \ a = \frac{1}{1+h^2}\varphi^-, \quad b = -\frac{h}{1+h^2}\varphi^-, \quad \psi^+ \geqslant 0, \quad \varphi^- < 0.$$

The above-mentioned cases exhaust all the possible relations between the quantities φ^+, φ^- ψ^+, and ψ^- and lead to formula (3) for the coefficients a and b. (The numbers 1-4 in Fig. 3 refer to the variants for $h < 1$.) Theorem 1 is proved.

Remark 1. The constructed solution u is the projection of the element w from X onto K [7].

Remark 2 (smoothness of the solution). Formula (2) and the properties of the functions w and α imply that if $f \in H^n(\Omega) \times H^m(\Omega)$, n ($m \geqslant 0$), then $u \in H^{n+2}(\Omega) \times H^{m+4}(\Omega)$, and if $f \in C^n(\Omega) \times C^m(\Omega)$, then $u \in C^{n+2}(\Omega) \times C^{m+4}(\Omega)$.

Remark 3. After the displacement function u of the beam has been found, the other physical characteristics of the state of the beam can be determined from (2), for example:

— the strain ε or the stress $\sigma = \varepsilon = Du$: $\sigma_1 = Dw_1 + a$ and $\sigma_2 = Dw_2 + bD\alpha$ (σ_1 is a continuous function in Ω_0);

— the potential energy of the beam $\Pi(u) = 0.5\|u\|^2 - \langle f, u\rangle = -0.5\|u\|^2$.

Remark 4. Let $f_2 \equiv 0$; then $w_2 \equiv 0 \Longrightarrow \varphi^+ = \varphi^- = \psi^+ = \psi^- = [w_1]$. Therefore, only variant Nos. 1 and 2 for the values of a and b can be realized. In any case, we have $b = 0$, i.e., $u_2 \equiv 0$.

Let $f_1 \equiv 0$; then $w_1 \equiv 0$, $\varphi^+ = h^2\psi^+ = -\varphi^- = -h^2\psi^- = h[Dw_2]$ and variant Nos. 3 and 4 can be realized. This implies that $a = -h/(1+h^2)|[Dw_2]|$ and $b = h^2/(1+h^2)[Dw_2]$. In both cases, $a \neq 0$ if $[Dw_2] \neq 0$ and, consequently, $u_1 \not\equiv 0$. Thus, there are no vertical displacements in the absence of vertical loads; in the absence of horizontal loads, horizontal displacements can occur owing to vertical loads.

Example. Let $f_1(x) \equiv c_1$, $f_2(x) \equiv c_2$, and $y = 0.5$. Then, we have

$$w_1(x) = \begin{cases} (c_1/2)(-x^2 + x), & x \in (0; 0.5), \\ (c_1/2)(-(1-x)^2 + (1-x)), & x \in (0.5; 1), \end{cases}$$

$$w_2(x) = \begin{cases} (c_2/48)(2x^4 - 4x^3 + 3x^2), & x \in (0; 0.5), \\ (c_2/48)(2(1-x)^4 - 4(1-x)^3 + 3(1-x)^2), & x \in (0.5; 1). \end{cases}$$

We find that

$$Dw_2(x) = \begin{cases} (c_2/24)(4x^3 - 6x^2 + 3x), & x \in (0; 0.5), \\ (c_2/24)(-4(1-x)^3 + 6(1-x)^2 - 3(1-x)), & x \in (0.5; 1), \end{cases}$$

$$[w_1] = 0, \quad [Dw_2] = -\frac{c_2}{24}, \quad \varphi^+ = -\frac{c_2 h}{24}, \quad \varphi^- = \frac{c_2 h}{24}, \quad \psi^+ = -\frac{c_2}{24h}, \quad \psi^- = \frac{c_2}{24h}.$$

Let $c_2 \geqslant 0$; then $\varphi^+ \leqslant 0$ and $\psi^- \geqslant 0$ and, consequently, $a = -\dfrac{c_2 h}{24(1+h^2)}$ and $b = -\dfrac{c_2 h^2}{24(1+h^2)}$.

If $c_2 \leqslant 0$, then $\varphi^- \leqslant 0$ and $\psi^+ \geqslant 0$ and $a = \dfrac{c_2 h}{24(1+h^2)}$ and $b = -\dfrac{c_2 h^2}{24(1+h^2)}$.

Thus, we have

$$u_1(x) = \begin{cases} \frac{1}{2}\Big(- c_1 x^2 + \Big(c_1 - \frac{|c_2| h}{12(1+h^2)}\Big) x\Big), & x \in (0; 0.5), \\ \frac{1}{2}\Big(- c_1 (1-x)^2 + \Big(c_1 + \frac{|c_2| h}{12(1+h^2)}\Big)(1-x)\Big), & x \in (0.5; 1), \end{cases}$$

$$u_2(x) = \begin{cases} \frac{c_2}{48}\Big(2x^4 - 4x^3 + \Big(3 - \frac{h^2}{1+h^2}\Big)x^2\Big), & x \in (0; 0.5), \\ \frac{c_2}{48}\Big(2(1-x)^4 - 4(1-x)^3 + \Big(3 - \frac{h^2}{1+h^2}\Big)(1-x)^2\Big), & x \in (0.5; 1). \end{cases}$$

It is noteworthy that $[u_1] = h|c_2|/(12(1+h^2))$.

Optimal Control of the Cut. The found solution (2) of the problem (1) depends on the function w, which is constructed for a fixed cut y. We rewrite (2) in the form of a dependence on the function $s = (s_1, s_2)$ that is continuous in Ω_0. For this purpose, we determine $s \in (H^2(\Omega_0) \cap H_0^1(\Omega_0)) \times (H^4(\Omega_0) \cap H_0^2(\Omega_0))$ as a solution of the boundary-value problem

$$-D^2 s_1 = f_1, \quad D^4 s_2 = f_2 \quad \text{in} \quad \Omega_0, \qquad s_1 = s_2 = Ds_2 = 0 \quad \text{for} \quad x = 0\ 1.$$

The function s is a function that describes the displacements of the points on the middle line of the beam without a cut. We introduce the function $\beta \in C^\infty(\omega)$:

$$\beta(x) = \begin{cases} (x^3 - 3yx^2)/6, & x \in (0; y-0), \\ ((x-1)^3 - 3(y-1)(x-1)^2)/6, & x \in (y+0; 1). \end{cases}$$

Its properties are as follows: $D^2\beta = x - y$, $D^3\beta \equiv 1$, $D^4\beta \equiv 0$ in Ω, $[D\beta] = y - 0.5$, and $\beta = D\beta = 0$ for $x = 0$ and 1. For convenience, we introduce the following notation: $d_1 = Ds_1(y)$, $d_2 = D^2 s_2(y)$, $d_3 = D^3 s_2(y)$, and $\Delta = d_2 - (y - 0.5)\, d_3$. Henceforth, the dependence of these quantities on y is not indicated.

Lemma 2. *The function w can be represented in the form*

$$w_1 = s_1 - d_1 D\alpha, \qquad w_2 = s_2 - d_2\alpha - d_3\beta.$$

Proof. By virtue of the properties of the functions α and β, we have

$$-D^2 w_2 = -D^2 s_1 + d_1 D^3\alpha = f_1, \quad D^4 w_2 = D^4 s_2 - d_2 D^4\alpha - d_3 D^4\beta = f_2,$$

$$Dw_1(y) = Ds_1(y) - d_1 D^2\alpha(y) = d_1 - d_1 = 0,$$

$$D^2 w_2(y) = D^2 s_2(y) - d_2 D^2\alpha(y) - d_3 D^2\beta(y) = d_2 - d_2 = 0,$$

$$D^3 w_2(y) = D^3 s_2(y) - d_2 D^3\alpha(y) - d_3 D^3\beta(y) = d_3 - d_3 = 0.$$

Lemma 2 is proved.

Theorem 2. *The function $u \in (H^2(\Omega) \times H^4(\Omega)) \cap X$ that is defined by the formula*

$$u = s - \eta, \qquad \eta = (AD\alpha, B\alpha + d_3\beta), \tag{4}$$

where

$$(A,B)=\begin{cases}(d_1,d_2), & d_1+h\Delta\geqslant 0,\quad d_1-h\Delta\geqslant 0,\\ \left(0,\left(y-\frac{1}{2}\right)d_3\right), & d_1+\frac{1}{h}\Delta<0,\quad d_1-\frac{1}{h}\Delta<0,\\ \frac{h}{1+h^2}\left(hd_1+\Delta, d_1+\frac{1}{h}d_2+h\left(y-\frac{1}{2}\right)d_3\right), & d_1+\frac{1}{h}\Delta\geqslant 0,\quad d_1-h\Delta<0,\\ \frac{h}{1+h^2}\left(hd_1-\Delta, -d_1+\frac{1}{h}d_2+h\left(y-\frac{1}{2}\right)d_3\right), & d_1-\frac{1}{h}\Delta\geqslant 0,\quad d_1+h\Delta<0,\end{cases} \tag{5}$$

is the solution of the variational inequality (1).

Proof. Theorem 2 can be proved using Lemma 2 and formulas (2) and (3). However, we find a proof similar to that for Theorem 1. To this end, we check whether the conditions for Lemma 1 are fulfilled:

$$-D^2u_1=-D^2s_1+AD^3\alpha=f_1\quad\text{in}\quad\Omega,\qquad D^4u_2=D^4s_2-BD^4\alpha-d_3D^4\beta=f_2\quad\text{in}\quad\Omega,$$

$$[Du_1]=-A[D^2\alpha]=0,\qquad [D^2u_2]=-B[D^2\alpha]-d_3[D^2\beta]=0,$$

$$D^3u_2(y)=d_3-BD^3\alpha(y)-d_3D^3\beta(y)=d_3-d_3=0.$$

Inasmuch as $Du_1(y)=d_1-A$, $D^2u_2(y)=d_2-B$, $[u_1]=A$, and $[Du_2]=B-(y-1/2)d_3$, the four conditions of Lemma 1 take the form

$$\left(d_1+\frac{1}{h}d_2-\left(A+\frac{1}{h}B\right)\right)\left(A+hB-h\left(y-\frac{1}{2}\right)d_3\right)=0,$$

$$\left(d_1-\frac{1}{h}d_2-\left(A-\frac{1}{h}B\right)\right)\left(A-hB+h\left(y-\frac{1}{2}\right)d_3\right)=0,$$

$$d_1+\frac{1}{h}d_2-\left(A+\frac{1}{h}B\right)\leqslant 0,\qquad d_1-\frac{1}{h}d_2-\left(A-\frac{1}{h}B\right)\leqslant 0,$$

$$A+hB-h\left(y-\frac{1}{2}\right)d_3\geqslant 0,\qquad A-hB+h\left(y-\frac{1}{2}\right)d_3\geqslant 0.$$

The following four variants are possible:

1) $d_1+\frac{1}{h}d_2-\left(A+\frac{1}{h}B\right)=0,\qquad d_1-\frac{1}{h}d_2-\left(A-\frac{1}{h}B\right)=0,$

$$A+hB-h\left(y-\frac{1}{2}\right)d_3\geqslant 0,\qquad A-hB+h\left(y-\frac{1}{2}\right)d_3\geqslant 0;$$

2) $d_1+\frac{1}{h}d_2-\left(A+\frac{1}{h}B\right)<0,\qquad d_1-\frac{1}{h}d_2-\left(A-\frac{1}{h}B\right)<0,$

$$A+hB-h\left(y-\frac{1}{2}\right)d_3=0,\qquad A-hB+h\left(y-\frac{1}{2}\right)d_3=0;$$

3) $d_1+\frac{1}{h}d_2-\left(A+\frac{1}{h}B\right)=0,\qquad d_1-\frac{1}{h}d_2-\left(A-\frac{1}{h}B\right)<0,$

$$A+hB-h\left(y-\frac{1}{2}\right)d_3\geqslant 0,\qquad A-hB+h\left(y-\frac{1}{2}\right)d_3=0;$$

4) $d_1+\frac{1}{h}d_2-\left(A+\frac{1}{h}B\right)<0,\qquad d_1-\frac{1}{h}d_2-\left(A-\frac{1}{h}B\right)=0,$

$$A+hB-h\left(y-\frac{1}{2}\right)d_3=0,\qquad A-hB+h\left(y-\frac{1}{2}\right)d_3\geqslant 0.$$

Solving these equations for A and B, we obtain formula (5). Theorem 2 is proved.

Remark 5. One can show that $[u_1]=0$ when $d_1\leqslant -(1/h)|\Delta|$. Moreover, $(A,B)=(0,(y-0.5)\,d_3)$ and, consequently, $u=(s_1,s_2-d_3((y-0.5)\alpha+\beta))$.

Remark 6. It follows from Remark 5 that $u = s$ when $d_3 = 0$ and $d_1 \leqslant -(1/h)|d_2|$. Obviously, the solution u is a continuous function in Ω_0.

We regard the coefficients d_1, d_2, d_3 Δ, A, and B as the functions of y and formulate the problem of optimal control

$$\inf_{0<y<1}[u_1]. \tag{6}$$

Problem (6) is interpreted as a problem of determining a cut that ensures the minimum opening [3]. Since $[u_1] = A$, the expression (6) is equivalent to $\inf\limits_{0<y<1} A$.

Theorem 3. *Let $f \in (C(\Omega_0))^2$; then the extremum of problem* (6) *can occur only at the points $y \in \Omega_0$ for which one of the following statements holds*:

1) $f_1(y) = 0$; $\quad d_1 \geqslant h|\Delta|$;

2) $d_1 \leqslant \dfrac{1}{h}|\Delta|$;

3) $f_1(y) = -\dfrac{1}{h}\Big(y - \dfrac{1}{2}\Big) f_2(y), \qquad d_1 < h\Delta, \qquad d_1 \geqslant -\dfrac{1}{h}\Delta$;

4) $f_1(y) = \dfrac{1}{h}\Big(y - \dfrac{1}{2}\Big) f_2(y), \qquad d_1 < -h\Delta, \qquad d_1 \geqslant \dfrac{1}{h}\Delta$;

5) $y = 0, \ 1$;

6) $d_1 = h|\Delta|$.

The infimum is equal to zero in Statement 2, *and it is not attained in Statement* 5.

Proof. Since $f \in (C(\Omega_0))^2$, we have $s \in C^2(\Omega_0) \times C^4(\Omega_0)$. It follows from (5) that A is a continuous function in Ω_0 which has the piecewise-continuous derivative and, possibly, discontinuities at the points y where the condition $d_1 = h|\Delta|$ or the condition $d_1 = (1/h)|\Delta|$ holds. Consequently, its extrema can be attained at the ends of the segment Ω_0 (Statement 5) or at the discontinuity points of the derivative (Statement 6 and $d_1 = (1/h)|\Delta|$ in Statement 2), or at the points y where $dA/dy = 0$. We calculate the derivatives

$$\frac{d}{dy} d_1 = \frac{d}{dy} Ds_1(y) = D^2 s_1(y) = -f_1(y),$$

$$\frac{d}{dy}\Delta = \frac{d}{dy}\Big(D^2 s_2(y) - \Big(y - \frac{1}{2}\Big) D^3 s_2(y)\Big) = -\Big(y - \frac{1}{2}\Big) f_2(y).$$

Substituting these values into formula (5) for the function A, we obtain Statements 1–4 of Theorem 3. Moreover, it follows from Remark 5 that $A = 0$ when $d_1 \leqslant (1/h)|\Delta|$. Theorem 3 is proved.

Example. Let $f_1(x) \equiv 1$, $f_2(x) \equiv 1$, and h be a small quantity: $h^2 < 1/12$. Consequently, s has the form

$$s_1(x) = 0.5(x - x^2), \qquad s_2(x) = (x^2 - 2x^3 + x^4)/24.$$

We find that $d_1 = 0.5 - y$, $d_2 = (1 - 6y + 6y^2)/12$, $d_3 = -0.5 + y$, and $\Delta = -1/6 + y/2 - y^2/2$. Note that $\Delta < 0$. Further

$$\begin{aligned} d_1 + h\Delta &= -\frac{h}{2}y^2 + \Big(\frac{h}{2} - 1\Big)y - \frac{h}{6} + \frac{1}{2}, \qquad & d_1 - h\Delta &= \frac{h}{2}y^2 - \Big(\frac{h}{2} + 1\Big)y + \frac{h}{6} + \frac{1}{2}, \\ d_1 + \frac{1}{h}\Delta &= -\frac{1}{2h}y^2 + \Big(\frac{1}{2h} - 1\Big)y - \frac{1}{6h} + \frac{1}{2}, \qquad & d_1 - \frac{1}{h}\Delta &= \frac{1}{2h}y^2 - \Big(\frac{1}{2h} + 1\Big)y + \frac{1}{6h} + \frac{1}{2}. \end{aligned} \tag{7}$$

We test all the statements of Theorem 3.

1. $f_1(y) \neq 0$.

2. Solving the quadratic equations $d_1 + (1/h)\Delta = 0$ and $d_1 - (1/h)\Delta = 0$, we infer that the discriminant $1 - 1/(12h^2) < 0$ and, consequently, $d_1 + (1/h)\Delta < 0$ and $d_1 - (1/h)\Delta > 0$ for any y. Thus, Statement 2 cannot hold.

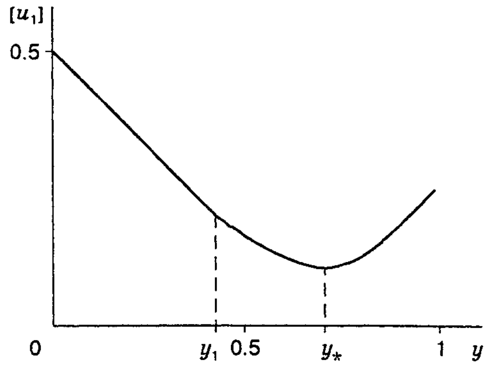

Fig. 4

3. We have $y = 1/2 - h$ and $d_1 + (1/h)\Delta < 0$, and the conditions are not satisfied.

4. Let $y_* = 1/2 + h$. We always have $d_1 - (1/h)\Delta > 0$. The condition $d_1 + h\Delta < 0$ remains to be checked for y_*. We find that $(d_1 + h\Delta)\big|_{y_*} = -(h/2)(h^2 + 25/12) < 0$ and $(d_1 - (1/h)\Delta)\big|_{y_*} = -h/2 + 1/(24h)$. Hence,

$$A\Big|_{y_*} = \frac{h^2}{1+h^2}\Big(d_1 - \frac{1}{h}\Delta\Big)\Big|_{y_*} = \frac{h(1-12h^2)}{24(1+h^2)}.$$

5. We check whether $y \to 0, 1$. For $y = 0$, we have $d_1 = 1/2$, $\Delta = -1/6$, and $d_1 > h|\Delta|$; therefore,

$$A\Big|_0 = 1/2 > A\Big|_{y_*}.$$

For $y = 1$, we have $d_1 = -1/2$, $\Delta = -1/6$, $d_1 + h\Delta < 0$, and $d_1 - (1/h)\Delta > 0$; hence

$$A\Big|_1 = \frac{h^2}{1+h^2}\Big(-\frac{1}{2} + \frac{1}{6h}\Big) = \frac{h(1-3h)}{6(1+h^2)} > A\Big|_{y_*}.$$

6. We assume that $d_1 + h\Delta = 0$. Solving the corresponding quadratic equation (7) for y, we obtain $y_1 = 1/2 - (1-K)/h$ and $K^2 = 1 - h^2/12$. Then, we calculate $A\big|_{y_1} = d_1\big|_{y_1} = 1/2 - y_1 = (1-K)/h > A\big|_{y_*}$. Assuming that $d_1 - h\Delta = 0$, we obtain the root $y_2 = 1/2 + (1-K)/h$. Since $(d_1 + h\Delta)\big|_{y_2} < 0$ and $(d_1 + (1/h)\Delta)\big|_{y_2} < 0$, the equality $d_1 - h\Delta = 0$ fails.

Thus, for $f_1 = f_2 \equiv 1$ and $0 < h < 1/(2\sqrt{3})$, at the point $y_* = 1/2 + h$ the minimum of (6)

$$\inf_{0<y<1}[u_1] = \frac{h(1-12h^2)}{24(1+h^2)}$$

is attained for

$$u_1 = s_1 - \frac{h(1-12h^2)}{24(1+h^2)} D\alpha, \quad u_2 = s_2 - \frac{24h^4 + 36h^2 - 1}{24(1+h^2)}\alpha - h\beta.$$

In this example, we searched for the extremum cut without finding the solution (4). Now we find an explicit expression for the coefficient A depending on $y \in \Omega_0$. Solving the quadratic equations (7), we obtain $d_1 + h\Delta \geqslant 0$ and $d_1 - h\Delta \geqslant 0$ when $y \in (0, y_1)$. In this interval, we have $A = 1/2 - y$. For $y \in (y_1, 1)$, we obtain $d_1 + h\Delta < 0$ and $d_1 - (1/h)\Delta > 0$ and, consequently, $A = (h^2/(1+h^2))(d_1 - (1/h)\Delta) = (h/(1+h^2))(y^2/2 - (1/2+h)y + 1/6 + h/2)$. Thus, we have

$$A = \begin{cases} 1/2 - y, & y \in (0, y_1), \\ \dfrac{h}{1+h^2}\Big(\dfrac{1}{2}y^2 - \Big(\dfrac{1}{2} + h\Big)y + \dfrac{1}{6} + \dfrac{h}{2}\Big), & x \in (y_1, 1). \end{cases}$$

Figure 4 shows the plot of the function $[u_1] = A$ versus y. Indeed, the condition $dA/dy = 0$ yields $y_* = 1/2 + h$, which is the minimum point of (6).

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